UDC 62-50

OPTIMAL ESTIMATES OF THE STATE OF A SYSTEM AND CERTAIN PROBLEMS OF CONTROL BY EQUATIONS WITH TIME LAG

PMM Vol. 41, № 3, 1977, pp. 446-456

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(Received October 6, 1976)

Problems of filtration and of inter- and extrapolation are considered in the case when the observed and the nonobserved components of a process are defined by linear stochastic differential equations with time lag. Analysis of such problems is reduced with the use of the principle of duality [1] to that of controllable systems with time lag. Optimal control of these is synthesized, conditions of existence and uniqueness of solutions of the Bellman equation are established, and exact solutions of the latter are presented.

A solution of the problem of extrapolation is readily obtained from that of the problem of filtration, while the analysis of that of interpolation is similar to the problem of filtration [2]. Because of this we only present the proof of formulas of optimal filtration and indicate the modifications that are necessary to these in their application to extra- and interpolation.

Let us assume that the Ito system of stochastic differential equations

$$dx(t) = (A(t)\,x(t) + B(t)x(t-h_1))\,dt + \sigma_1(t)\,d\xi_1(t) \tag{1.1}$$

$$0 \leqslant t \leqslant T, \quad x(s) = 0, \quad s < 0, \quad x(0) = x_0$$

defines some nonobserved random proces $x(t)$ and that the observable process $y(t)$ satisfies the relationship

$$dy(t) = [g(t)\,x(t-h_2) + k(t)\,x(t-h_3)]\;dt + \sigma_2(t)\,d\xi_2(t) \tag{1.2}$$

$$y(0) = 0$$

where $T \geqslant 0$ is an arbitrary fixed instant of time and $x(t)$belongs to an n-dimensional Euclidean space and E_n, a $y(t) \in$. The elements of matrices $A(t)$ $g(t)$, $k(t)$, $\sigma_1(t)$, and $\sigma_2(t)$ are assumed to be piece-wise continuous functions, and the elements of matrix $B(t)$ to be piece-wise differentiable. The random vector x_0 is independent of the mutually independent multidimensional standard Wiener processes $\xi_1(t)$ and $\xi_2(t)$ and has normal distribution with zero mathematical expectation and correlation matrix H. The dimensions of vectors ξ_1 and ξ_2 are arbitrary. Matrix $N_1(t) = \sigma_2(t)\ \sigma_2'(t)$ is uniformly positive definite on $[0, T]$ and the prime indicates transposition. The constants h_1, h_2, and h_3 are nonnegative and $h_2 \leqslant h_3$.

Note that systems of form (1.1), (1.2) are considered here only for simplicity of notation. The whole of the further reasoning is valid for equations with several discrete and, also, distributed time-lag.

On the stated assumptions the existence and uniqueness of solution of Eq. (1.1) is that given in [3]. We denote by $m(T)$, $D(T)$ respectively, the conditional mathematical expectation and the correlation matrix of the random vector $x(T)$with condition $y(t)$, $0 \leqslant t \leqslant T$. It is shown in [3] that $m(T)$is the best estimate, in the sense of the square mean, of $x(T)$,and $D(T)$ is the correlation matrix of vector $x(T) - m(T)$. The expressions for $m(T)$ and $D(T)$ represent formulas of optimal filtration.

We introduce in the analysis n vectors $\alpha_i(t) \in E_n (i = 1, 2, \ldots, n)$ each of which is determined along segment $[0, T]$ by the solution of the system of equations

$$\alpha_i^{\cdot}(t) = -A'(t)\alpha_i(t) - B'(t+h_1)\alpha_i(t+h_1) + g'(t+h_2)u_i(t+h_2) + k'(t+h_3)u_i(t+h_3) \quad (1.3)$$

in which the control $u_i(t)$ must minimise along the trajectories of system (1.3) the quadratic functional

$$J(u_i) = \alpha_i'(0)H\alpha_i(0) + \int_0^T [u_i'(t)N_1(t)u_i(t) + \alpha_i'(t)N_2(t)\alpha_i(t)]\,dt, \quad N_2 = \sigma_1\sigma_1' \quad (1.4)$$

Problem (1.3), (1.4) is solved for the following initial conditions: $\alpha_i(t) = 0$ when $t > T$, the i-th component of vector $\alpha_i(T)$ is equal unity and all remaining are zero: $u_i(t) = 0$ when $t > T$.

It is furthermore assumed that $B(t)$, $g(t)$ and $k(t)$ are zero outside of segment $[0, T]$.

Note that the equation of filtration when $g(t) = 0$ and system (1.3), (1.4) does not, actually, contain any timelag in the control, were obtained in [4]. An essential feature of problem (1.3), (1.4) considered here is the presence of timelag in phase coordinates and in the control, which obviously is interesting in itself.

We denote by $m_i(T)$ the i-th component of vector $m(T)$ and by $d_{ij}(T)$ the elements of matrix $D(T)$; $i, j = 1,2,\ldots, n$. Then, using the modified Kalman's principle of duality between the control and observation [1], for systems with timelag appearing in [4] we conclude that

$$m_i(T) = \int_0^T u_i'(t)\,dy(t) \quad (1.5)$$

$$d_{ii}(T) = J(u_i) = M\left[x_i(T) - \int_0^T u_i'(t)\,dy(t)\right]^2$$

$$d_{ij}(T) = M\left[x_i(T) - \int_0^T u_i'(t)\,dy(t)\right]\left[x_j(T) - \int_0^T u_j'(t)\,dy(t)\right]$$

where M is the symbol of mathematical expectation.

Note that the principle of duality between the control and observation was developed in [5, 6] in a minimax formulation for ordinary systems, and in [7] for systems with timelag.

To determine the remaining elements $d_{ij}(T)$ it is sufficient to solve the optimal problem (1.3), (1.4) by selecting as initial conditions at $t = T$ vectors α_i (T) whose ij -th components are unity and the remaining are zero.

Thus the construction of the optimal filter on the basis of (1.4) and (1.5) is reduced to the analysis of the control problem (1.3), (1.4). Let us carry out some simplifying transformations in (1.3) and (1.4). We substitute variables $t \to T - t$. Then retaining previous notation for all functions and omitting subscript i, we obtain

$$\dot{\alpha}(t) = A'(T-t)\alpha(t) + B'(T-t+h_1)\alpha(t-h_1) - g'(T-t+h_2)u(t-h_2) + k'(T-t+h_3)u(t-h_3) \quad (1.6)$$

$$J(u) = \alpha'(T)H\alpha(T) + \int_0^T [u'(t)N_1(t)u(t) + \alpha'(t)N_2(t)\alpha(t)]\,dt \quad (1.7)$$

The initial conditions of problem (1.6), (1.7) for $t \leqslant 0$ are, obviously, the same as the corresponding conditions of problem (1.3), (1.4) when $t \geqslant T$. We further assume that in (1.6) the matrix $A(t) \equiv 0$. This can alwavs be obtained by the substitution of variables $\alpha(t) \to z(0, t)\alpha(t)$, where $z(s, t)$ is the fundamental solution of system (1.6) when $B(t) \equiv 0$, $g(t) \equiv 0$ and $k(t) \equiv 0$. As the result of this and some simple transformations system (1.6) can be presented in the form

$$\alpha^{\cdot}(t) = B(t)\alpha(t-h_1) + g(t)u(t) + k(t)u(t-h) \quad (1.8)$$

The quality criterion retains its previous form (1.7), and the parameters of problem (1.7), (1.8) are readily expressed in terms of relationships of the input control problem (1.6(, (1.7), as described above.

Problem (1.8), (1.7) is solved for initial conditions $\alpha = a(\tau)$ $(\tau \leqslant 0)$, $u = b(\tau)$,' $(\tau < 0)$; where $a(\tau)$ and $b(\tau)$ are some measurable bounded functions. The reasoning and the methods used here in the analysis of this problem are in several instances similar to those used in [8]. Hence in further synthesis we shall dwell only on distinctive features. Note that the establishment of the necessary conditions of optimality of systems with timelag in the control was dealt in [9-11].

We denote by $V_0(t, \alpha, \alpha(t+\tau), u(t+\rho))$ the minimum value of functional

$$V(t, \alpha, \alpha(t+\bar{\tau}), u(t+\rho)) = \alpha'(T)H\alpha(T) + \int_t^T [u'(s)N_1(s)u(s) + \alpha'(s)N_2(s)\alpha(s)]\,ds \quad (1.9)$$

where $\alpha = \alpha(t)$; $\alpha(t+\tau)$, $u(t+\rho)$ $(-h_1 \leqslant \bar{\tau} < 0,\ -h \leqslant \rho \leqslant 0)$ are segments of the trajectory of system (1.7) and the control in intervals $(t - h_1, t)$ and $(t - h, t)$ respectively. It can be shown [12] that if functionals V are considered to be functions of form $v(t, \alpha)$, it is sufficient for the determination of optimal values of the functional of the criterion of quality and control to solve the Bellman equation

$$\min_{u \in E_l} \left\{\frac{dv(t, \alpha)}{dt} + u'N_1(t)u + \alpha'N_2(t)\alpha\right\} = 0 \quad (1.10)$$

where dv / dt is the total derivative along the trajectories of system (1.7) with control u [13].

We seek the solution of Eq. (1.10) of the form

$$v(t, \alpha) = V_0(t, \alpha, \alpha(t+\bar{\tau}), u(t+\rho)) = \alpha'(t) P_1(t)\alpha(t) + \tag{1.11}$$
$$\alpha'(t)\int_{-h_1}^{0} P_2(t, s)\alpha(t+s)\,ds + \int_{-h_1}^{0} \alpha'(t+s) P_2'(t, s)\alpha(t)\,ds +$$
$$\alpha'(t)\int_{-h}^{0} P_4(t, r)u(t+r)\,dr + \int_{-h}^{0} u'(t+r) P_4'(t, r)\alpha(t)\,dr +$$
$$\int_{-h_1}^{0}\int_{-h_1}^{0} \alpha'(t+s) P_3(t, s, s_1)\alpha(t+s_1)\,ds_1\,ds +$$
$$\int_{-h_1}^{0}\int_{-h}^{0} \alpha'(t+s) P_5(t, s, r) u(t+r)\,dr\,ds +$$
$$\int_{-h_1}^{0}\int_{-h}^{0} u'(t+r) P_5'(t, s, r)\alpha(t+s)\,dr\,ds +$$
$$\int_{-h}^{0}\int_{-h}^{0} u'(t+r) P_6(t, r, r_1) u(t+r_1)\,dr\,dr_1$$

where P_i $(i = 1, 2, \ldots, 6)$ are some matrices with fairly smooth elements and $P_0(t)$ is nonnegative definite, while condition $P_i'(t, s, r) = P_i(t, r, s)$ is satisfied when $i = 3,6$.

Substituting (1.8) and (1.11) into (1.10) we find, after transformation, that the synthesis of optimal control $u_0(t) = u_0(t, \alpha, \alpha(t+\bar{\tau}), u(t+\rho))$ is determined by the relationship

$$u_0(t) = -N_1^{-1}(t)([g'(t) P_1(t) + P_4'(t, 0)]\alpha(t) + \tag{1.12}$$
$$\int_{-h_1}^{0} [g'(t) P_2(t, s) + P_5'(t, s, 0)]\alpha(t+s)\,ds +$$
$$\int_{-h}^{0} [g'(t) P_4(t, r) + P_6(t, 0, r)] u_0(t+r)\,dr$$

which specifies such synthesis in the form of a functional of the obtained trajectory and control. Note that equality (1.12) represents the Volterra integral equation in $u_0(t)$. Hence by determining from it $u_0(t)$ by conventional procedures it is possible to obtain the optimal control in the form of a functional that depends only on phase coordinates.

Combining (1.9)-(1.12) and using the method of indeterminate coefficients, for the determination of the unknown functions P_i we obtain the system of equations in partial derivatives

$$P_1^{\bullet}(t) + P_2(t, 0) + P_2'(t, 0) + N_2(t) = [P_1(t) g(t) + \tag{1.13}$$
$$P_4(t, 0)] N_1^{-1}(t) [g'(t) P_1(t) + P_4'(t, 0)]$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial s}\right)P_2(t,s)+P_3(t,0,s)=$$
$$[P_1(t)g(t)+P_4(t,0)]N_1^{-1}(t)[g'(t)P_2(t,s)+P_5'(t,s,0)]$$
$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial s}-\frac{\partial}{\partial s_1}\right)P_3(t,s,s_1)=$$
$$[P_2'(t,s)g(t)+P_5(t,s,0)]N_1^{-1}(t)[g'(t)P_2(t,s_1)+P_5'(t,s_1,0)]$$
$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial r}\right)P_4(t,r)+P_5(t,0,r)=$$
$$[P_1(t)g(t)+P_4(t,0)]N_1^{-1}(t)[g'(t)P_4(t,r)+P_6(t,0,r)]$$
$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial s}-\frac{\partial}{\partial r}\right)P_5(t,s,r)=$$
$$[P_2'(t,s)g(t)+P_5(t,s,0)]N_1^{-1}(t)[g'(t)P_4(t,r)+P_6(t,0,r)]$$
$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial r}-\frac{\partial}{\partial r_1}\right)P_6(t,r,r_1)=$$
$$[P_4'(t,r)g(t)+P_6(t,r,0)]N_1^{-1}(t)[g'(t)P_4(t,r_1)+P_6'(t,r_1,0)]$$
$$0\leqslant t\leqslant T,\quad -h_1\leqslant s,s_1\leqslant 0,\quad -h\leqslant r,r_1\leqslant 0$$

The system of boundary conditions is obtained in the same way. For any $h_1 < s$, $s_1 \leqslant 0$, $-h < r$, and $r_1 \leqslant 0$.

$$P_1(T)=H,\ P_2(T,s)=P_3(T,s,s_1)=P_4(T,r)=P_5(T,s,r)=P_6(T,r,r_1)=0 \tag{1.14}$$

For $t < T$

$$\begin{aligned}
&P_1(t)k(t)-P_4(t,-h)=0\\
&k'(t)P_2(t,s)-P_5'(t,s,-h)=0\\
&B'(t)P_1(t)-P_2'(t,-h_1)=0\\
&2B'(t)P_2(t,s)-P_3(t_3-h_1,s)-P_3'(t,s,-h_1)=0\\
&B'(t)P_4(t,r)-P_5(t,-h_1,r)=0\\
&2k'(t)P_4(t,r)-P_6(t,-h,r)-P_6'(t,r,-h)=0
\end{aligned} \tag{1.15}$$

Thus, when the optimal value of functional (1.9) is of the form (1.11) and coefficients P_i are reasonably smooth, these functions represent (almost everywhere) the unique solution of problem (1.13)-(1.15). Conversely, if there exists a solution of problem (1.13)-(1.15) it is unique for almost all values of arguments and determines the optimal control, the system trajectory and the optimal value of the quality criterion by formulas (1.8), (1.11), and (1.12).

Proof of the existence of the solution of the system of Eqs. (1.13)-(1.15) is carried out by establishing the algorithm of successive approximations which was proposed for other problems in [12].

Let $u_1 = u_1(t)$ (the first approximation of the optimal control) satisfy for $t \geqslant 0$ the relation

$$u_1(t) = q_1(t)\alpha_1(t) + \int_{-h_1}^{0} q_2(t, s)\alpha_1(t+s)\,ds + \int_{-h}^{0} q_3(t, r)u_1(t+r)\,dr \tag{1.16}$$

where matrices $q_1(t)$, $q_2(t, s)$ and $q_3(t, r)$ are bounded functions piecewise continuously differentiable with respect to t, s, and r; $u_1(\tau) = \omega(\tau)$; when $\tau < 0$ and $\omega(\tau)$ is some specified measurable bounded function, and $\alpha_1(t)$ is the solution of (1.8) when $u = u_1$. Let us take two arbitrary instants of time t, and s, with $s > t$ and establish the formula for expressing $\alpha_1(s)$, and $u_1(s)$ in terms of $\alpha_1(t)$, and $u_1(\tau)$ for $\tau \leqslant t$. For this we shall consider formulas (1.8) and (1.16) as a system of $n + l$ equations for the determination of $n + l$ components of vector $z(t) = \{\alpha_1(t), u_1(t)\}$. It is not difficult to see that solutions of that system are the same as the solutions of the system formed by (1.8) and the equation of the form

$$\begin{aligned} u_1^{\cdot}(t) = {} & [q_1^{\cdot}(t) + q_2(t, 0)]\alpha_1(t) + [q_1(t)g(t) + q_3(t, 0)]u_1(t) + \\ & [q_1(t)B(t) - q_2(t, -h_1)]\alpha_1(t-h_1) + [q_1(t)k(t) - \\ & q_3(t, -h)]u_1(t-h) + \int_{-h_1}^{0}\left[\frac{\partial}{\partial t} - \frac{\partial}{\partial s}\right] q_2(t, s)\alpha_1(t+s)\,ds + \\ & \int_{-h}^{0}\left[\frac{\partial}{\partial t} - \frac{\partial}{\partial r}\right] q_3(t, r)u_1(t+r)\,dr \end{aligned} \tag{1.17}$$

which is obtained by differentiating term by term the right- and the left-hand sides of (1.16).

The boundary conditions for (1.17) are of the form

$$u_1 = \omega(\tau), \quad \tau < 0, \quad u_1(0) = q_1(0)\alpha_1(0) + \int_{-h_1}^{0} q_2(0, s)\alpha_1(s)\,ds + \int_{-h}^{0} q_3(0, r)\omega(r)\,dr$$

Combining (1.8) and (1.17) we obtain an ordinary system of $n + l$ differential equations with the deflecting argument

$$z^{\cdot}(t) = C(t)z(t) + \sum_{i=1}^{2} D_i(t)z(t-h_i) + \sum_{i=1}^{2}\int_{-h_i}^{0} G_i(t, \sigma)z(t+\sigma)\,d\sigma \tag{1.18}$$

$$(z(\tau) = \{a(\tau), \ u_1(\tau)\}, \quad \tau \leqslant 0)$$

where $h_2 = h$. Matrices $C(t)$ and $D_i(t)$ are defined by the coefficients of (1.8), (1.16), and (1.17) as follows:

$$C(t) = \begin{Vmatrix} 0 & g(t) \\ q_1^{\cdot}(t) + q_2(t, 0) & q_1(t)g(t) + q_3(t, 0) \end{Vmatrix}$$

$$D_1(t) = \begin{Vmatrix} B(t) & 0 \\ q_1(t)B(t) - q_2(t, -h_1) & 0 \end{Vmatrix}$$

$$D_2(t) = \begin{Vmatrix} 0 & k(t) \\ 0 & q_1(t)k(t) - q_3(t, -h) \end{Vmatrix}$$

Matrices $G_i(t, \sigma)$ $(i = 1, 2, -h_i \leqslant \sigma_i \leqslant 0)$ are of the form

$$G_1(t, \sigma_1) = \begin{Vmatrix} 0 & 0 \\ \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \sigma_1}\right) q_2(t, \sigma_1) & 0 \end{Vmatrix}$$

$$G_2(t, \sigma_2) = \begin{Vmatrix} 0 & 0 \\ 0 & \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \sigma_2}\right) q_3(t, \sigma_2) \end{Vmatrix}$$

By Cauchy's formula [14] the solution of system (1.18) satisfies for all $s \geqslant t$ the the relationship

$$z(s) = K(s, t)z(t) + \sum_{i=1}^{2} \int_{-h_i}^{0} K_i(s, t, \sigma) z(t+\sigma) d\sigma \tag{1.19}$$

$$K_i(s, t, \sigma) = K(s, t+\sigma+h_i) D_i(t+\sigma+h_i) + \int_{-h_i}^{\sigma} K(s, t+\sigma-\rho) G_i(t+\sigma-\rho, \rho) d\rho$$

where $K(s, t)$ is the fundamental solution of system (1.18). By virtue of (1.16) and because of the form of patrices of D_i and G_i formula (1.19) splits into the following two:

$$\alpha_1(s) = \beta_1(s, t)\alpha_1(t) + \int_{-h_1}^{0} \beta_2(s, t, \tau)\alpha_1(t+\tau) d\tau + \int_{-h}^{0} \beta_3(s, t, \sigma) u_1(t+\sigma) d\sigma \tag{1.20}$$

$$u_1(s) = \gamma_1(s, t)\alpha_1(t) + \int_{-h_1}^{0} \gamma_2(s, t, \tau)\alpha_1(t+\tau) d\tau + \int_{-h}^{0} \gamma_3(s, t, \sigma) u_1(t+\sigma) d\sigma$$

where β_i, and γ_i $(i = 1, 2, 3)$ are some matrices whose elements are piecewise continuously differentiable. Their expressions in terms of elements K, K_i and q_i can be obtained by the procedure indicated above. Formulas (1.20) determine control $u_1(s)$ and the related trajectory $\alpha_1(s)$ of system (1.8) for all $t \leqslant s \leqslant T$, if initial values $u(t+\rho)$ $(-h < \rho < 0)$, $\alpha(t+\tau)$ $(-h_1 < \tau < 0)$ and $\alpha(t)$ are arbitrarily specified.

Using formulas (1.20) we effect the substitution $\alpha(s) = \alpha_1(s)$ and $u(s) = u_1(s)$ in the right-hand side of (1.9) and denote the result of that substitution by V_1. After transformation we find that the functional $V_1 = V_1(t, \alpha, \alpha(t+\bar{\tau}), u(t+\rho))$ is of the form (1.11) with some coefficients P_{i1}, $i = 1, 2, \ldots, 6$. Considering V_1 as a function of arguments t and α, i.e. $V_1 = v_1(t, \alpha)$, where $\alpha = \alpha(t)$, we find that the derivative

$$\frac{dV_1}{dt} = \frac{\partial v_1}{\partial t} + \frac{\partial v_1}{\partial \alpha} \alpha^{\cdot}(t)$$

is a certain functional $l_1(t, \alpha(t), u(t), \alpha(t+\tau), u(t+\rho))$ and that for all $t \leqslant T$ the equality

$$l_1(t, \alpha_1(t), u_1(t), \alpha_1(t+\tau), u_1(t+\rho)) + u_1'(t) N_1(t) u_1(t) + \alpha_1'(t) N_2(t) \alpha_1(t) = 0 \tag{1.21}$$

is satisfied by construction.

Let us, now, determine $u_2 = u_2(t)$ (the second approximation of optimal control) so that condition

$$\min_{u \in R_l} \{l_1(t, \alpha, u, \alpha(t+\tau), u(t+\rho)) + u'N_1(t) u + \alpha' N_2(t) \alpha\} = l_1(t, \alpha, u_2(t), \alpha(t+\tau), u(t+\rho)) + u_2'(t) N_1(t) u_2(t) + \alpha' N_2(t) \alpha \tag{1.22}$$

is satisfied for any $t, \alpha, \alpha(t+\tau)$ and $u(t+\rho)$

It is seen from (1.22) that $u_2(t)$ satisfies relationships of the form (1.12), if in the latter u_2, α_2 and P_{i1} $(i = 1, 2, \ldots, 6)$ are, respectively, substituted for functions u_0, α and P_i and $\alpha_2 = \alpha_2(t)$ is the trajectory of system (1.8) that corresponds to control u_2. Thus $u_2(t)$ is determined by an equation similar to (1.16) and, consequently, all results obtained for the control $u_1(t)$, remain valid.

Applying a similar procedure we determine the sequence of controls $u_k(t) = u_k(t, \alpha, \alpha(t+\bar{\tau}), u(t+\rho))$ by relationships

$$\min_{u \in R_l} \left\{\frac{d}{dt} V_{k-1}(t) + u'N_1(t) u + \alpha'(t) N_2(t) \alpha(t)\right\} = \frac{d}{dt} V_{k-1}(t) + u_k'(t) N_1(t) u_k(t) + \alpha'(t) N_2(t) \alpha(t) \tag{1.23}$$

that are similar to (1.21) and (1.22).

The sequences of functional V_k is the result of substitution into the right-hand side of (1.9) of expressions for $u_k(s)$ and $\alpha_k(s)$, where $\alpha_k(s)$ is the solution of Eq. (1.8) for $u(s) = u_k(s)$. As in the case of V_1, and u_2, we establish that V_k is of the form (1.11) with certain coefficients P_{ik} $(i = 1, 2, \ldots, 6)$. Control $u_k(t)$ satisfies an equation of the form (1.12), if in the latter we set $u_0 = u_k$, $\alpha = \alpha_k$, and $P_i = P_{i,k-1}$. The coefficients of functionals u_k, and V_k have the same properties as the coefficients of first approximations, and for all k the equality

$$\frac{d}{dt} V_k + u_k'(t) N_1(t) u_k(t) + \alpha_k'(t) N_2(t) \alpha_k(t) = 0 \tag{1.24}$$

is valid in the sense of (1.21).

Owing to the arbitrariness of the setting of the system trajectories and of control in intervals $(t - h_1, t]$ and $(t - h, t)$ respectively, we obtain from (1.24) by the method of indeterminate coefficients, a system of linear equations in partial derivatives, which is satisfied by functions P_{ik}. The left-hand sides of these equations and the boundary conditions are of the form (1.13)-(1.15) in which P_{ik} is substituted for P_i and whose right-hand sides which can be represented as the product of matrices R $N_1^{-1}(t)\,S$ are replaced by

$$R_k N_1^{-1}(t)\, S_{k-1} + R_{k-1} N_1^{-1}(t)\, S_k - R_{k-1} N_1^{-1}(t)\, S_{k-1}$$

where R_k and S_k are binomials that correspond to R or S and are obtained by the substitution of P_{ik} for P_i.

The passing to limit $k \to \infty$ in formulas (1.23) and (1.24) and in the system of differential equations in functions P_{ik} is validated as in [8]. Hence the following theorem is established.

Theorem 1. If we assume that matrices $g(t)$ and $k(t)$ in (1.8), and $N_1(t)$ and $N_2(t)$ have piecewise continuous elements and $B(t)$ are piecewise continuously differentiable, the solution of the problem of optimal control of system (1.8) with the quality criterion (1.9) is represented in the form (1.11), (1.12), and the coefficients of optimal functionals u_0 and V_0 represent the unique solution of the system of Eqs. (1.13)-(1.15).

Note. When $k(t) = 0$ and $h = 0$ the statements of this theorem coincide with the results presented in [8]. We should also point out that the successive approximation P_{ik}, used in the proof of the theorem may be taken as the approximate solution of problem (1.13)-(1.15), and it is possible to establish as in [15] that for some constant c

$$\| P_i(t) - P_{ik}(t) \| \leqslant c^i / i!$$

2. Exact solutions. We present here formulas which provide the solution of the boundary value problems (1.13)-(1.15) on the additional ssumption that $N_2(t) \equiv 0$, and that matrices $B(t)$ and $k(t)$ are absolutely continuous. Note that when $N_2(t) \neq 0$ the problem (1.13)-(1.15) is generally not integrable even for controlled systems without timelag, i.e. for systems (1.8) with $h = h_1 = 0$.

We define matrix $\gamma(t)$ by the relationship

$$\gamma^{\cdot}(t) = -\gamma(t + h_1)\, B(t + h_1), \quad 0 \leqslant t \leqslant T$$
$$\gamma(T) = I, \quad \gamma(s) \equiv 0, \quad s > T$$

where I is a unit matrix.

A direct test will prove that problem (1.13)-(1.15) has almost everywhere the following solution:

$$P_1(t) = \gamma'(t)\, P(t)\, \gamma(t) \tag{2.1}$$
$$P_2(t, s) = \gamma'(t)\, P(t)\, \gamma(t + h_1 + s)\, B(t + h_1 + s)$$
$$P_3(t, s, s_1) = B'(t + h_1 + s)\, \gamma'(t + h_1 + s)\, P(t)\, \gamma(t + h_1 + s)\, B(t + h_1 + s)$$
$$P_4(t, r) = \gamma'(t)\, P(t)\, k_1(t + r + h)$$
$$P_5(t, s, r) = B'(t + h_1 + s)\, \gamma'(t + h_1 + s)\, P(t)\, k_1(t + r + h)$$
$$P_6(t, r, r_1) = k_1'(t + r + h)\, P(t)\, k_1(t + r_1 + h)$$
$$(t \leqslant T, \quad -h < s, s_1, r, r_1 \leqslant 0)$$

where

$$k_1(t+r+h) = \begin{cases} \gamma(t+r+h)\,k(t+r+h), & t+r+h \leqslant \min(T, \quad r+h) \\ 0, & t+r+h > \min(T, \quad r+h) \end{cases}$$

Moreover $P(t)$ satisfies the Bernoulli matrix equation $P^{\cdot}(t) = P(t) \quad G(t)\,N_1^{-1}(t)G'(t)\,P(t), \quad P(T) = H$

$$G(t) = \begin{cases} \gamma(t)\,g(t), & T-h < t \leqslant T \\ \gamma(t)\,g(t) + \gamma(t+h)\,k(t+h), & 0 \leqslant t \leqslant T-h \end{cases}$$

It can be verified that when $H > 0$

$$P(t) = \left[I + H\int_t^T G(s)\,N_1^{-1}(s)\,G'(s)\,ds\right]^{-1} H$$

Note that solution (2.1) of problem (1.13)-(1.15) was obtained for $k(t) \equiv 0$ in [16], where the method of derivation of formula (2.1) is also described.

3. The solution of the filtration problem derived above is based on its reduction by the use of the principle of duality to some problem of optimal control and the analysis of the latter. A similar algorithm is valid for problems of extra- and interpolation whose specific properties only appear in concrete form of the dual problem of optimal control.

Let us first consider in detail the problem of extrapolation. It consists of the derivation of the optimal estimate of process (1.1) at instant $\tau > T$ on condition that along segment $[0, T]$ the quantity (1.2) is observed. We denote by $g_0(t)$ and $k_0(t)$ the functions that coincide, respectively, with $g(t)$ and $k(t)$ for $0 \leqslant t \leqslant T$ and are zero when $t > T$.

Then we consider the subsidiary problem of filtration of vector $x(\tau)$, which satisfies formulas (1.1) on the basis of observation of process $y_0(t)$, which is defined by Eq.(1.2) in which $g_0(t)$ and $k_0(t)$ have been substituted for $g(t)$ and $k(t)$ Owing to the independence of x_0, ξ_1 and ξ_2 the solution of that subsidiary problem of filtration is also the solution of the extrapolation problem. Hence the double extrapolation of the optimal control problem is of the same form as in Sect. 1 with the substitution of τ for T and $g_0(t)$ and $k_0(t)$ for $g(t)$ and $k(t)$ respectively. Note, also, that the optimal control in this dual problem is zero when $t > T$ owing to the definition of functions g_0 and k_0.

Let us now turn to interpolation which consists of the optimal estimate of process (1. 1) at instant $\tau \in [0, T)$ on the basis of observation $y(t)$ along the segment $0 \leqslant t \leqslant T$, where $y(t)$ is defined by formulas (1.2). We consider the dual problem of optimal control of system

$$\alpha_i^{\cdot}(t) = -A'(t)\,\alpha_i(t) - B'(t+h_1)\,\alpha_i(t+h_1) + g'(t+h_2)\,u_i(t+h_2) + k'(t+h_3)\,u_i(t+h_3) - \delta_i(t-\tau_0) \tag{3.1}$$

$$\alpha_i(s) = 0, \; u_i(s) = 0, \; s \geqslant T$$

where $\delta_i(t)$ is a vector whose i-th component is the delta function in zero, and the remaining are zero. Equation (3.1) is understood in the sense of the related integral identity.

The minimizable functional $J(u_i)$ is of the form (1.4). As in the case of (1.5), we conclude that the optimal estimate of the i-th component of vector $x(\tau_0)$ is

$$\int_0^T u_i'(t)\,dy(t)$$

and

$$J(u_i) = M\left[x_i(\tau_0) - \int_0^T u_i'(t)\,dy(t)\right]^2$$

To solve the problem of synthesis of (3.1) and (1.4) we reduce system (3.1) to a form similar to (1.8)

$$\alpha^{\cdot}(t) = B(t)\,\alpha(t-h_1) + g(t)\,u(t) + k(t)\,u(t-h) + \delta_i(t-\tau_0) \tag{3.2}$$

The optimal value of the functional in the problem (3.2), (1.4) can be represented, as in the problem (1.8), (1.4), in the form $V_1 + V_2$, where V_1 has the form (1.11) and V_2 is determined by the following linear relation in u and α

$$\begin{aligned} V_2 = V_2(t, \alpha, \alpha(t+\bar{\tau}), u(t+\rho)) = P_7'(t)\alpha(t) + \alpha'(t)P_7(t) + \\ \int_{-h_1}^{0} P_8'(t,s)\alpha(t+s)\,ds + \int_{-h_1}^{0} \alpha'(t+s)P_8(t,s)\,ds + \\ \int_{-h}^{0} P_9'(t,r)u(t+r)\,dr + \int_{-h}^{0} u'(t+r)P_9(t,r)\,dr + P_{10}(t) \end{aligned} \tag{3.3}$$

Similarly to (1.12) the optimal control may be written in the form $u = u_1 + u_2$ where u_1 is determined by formula (1.12) and

$$u_2(t) = -N^{-1}(t)[g'(t)\,P_7(t) + P_9(t, 0)]$$

Functions $P_i,\ i = 7, 8, 9$ satisfy the equations

$$\begin{aligned} & P^{\cdot}_7(t) + P_8(t, 0) + P_1(t)\,\delta_i(t-\tau_0) = \\ & \quad [P_1(t)\,g(t) + P_4(t, 0)]N^{-1}(t)\,[g'(t)\,P_7(t) + P_9(t, 0)] \\ & \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial s}\right)P_8(t,s) + P_2'(t,s)\,\delta_i(t-\tau_0) = [P_2'(t,s)\,g(t) + \\ & \quad P_5(t,s,0)]\,N^{-1}(t)\,[g'(t)\,P_7(t) + P_9(t,0)] \\ & \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial r}\right)P_9(t,r) + P_4'(t,r)\,\delta_i(t-\tau_0) = \\ & \quad [P_4'(t,r)\,g(t) + P_5(t,r,0)]\,N^{-1}(t)\,[g'(t)\,P_7(t) + P_9(t,0)] \end{aligned} \tag{3.4}$$

which are understood in the sense of an integral identity.

Finally, $P_{10}(t)$ is determined by formula

$$P_{10}(t) = -P_7'(\tau_0)f_i(t-\tau_0) + \int_T^t [P_7'(s)g(s) + P_9'(s,0)]N^{-1}(s)[g'(s)P_7(s) + P_9(s,0)]\,ds \tag{3.5}$$

where vector $f_i(t-\tau_0)$ is zero when $t \neq \tau_0$; and when $t = \tau_0$ its i-th component is equal unity and the remaining vanish.

The boundary conditions for system (3.4) are of the form

$$\begin{aligned} &P_7(T) = P_8(T, s) = P_9(T, r) = 0 \qquad (-h_1 \leqslant s \leqslant 0, \; -h \leqslant r \leqslant 0) \\ &B'(t)P_7(t) - P_8(t, -h_1) = 0 \\ &k'(t)P_7(t) - P_9(t, -h) = 0, \quad 0 < t < T \end{aligned} \tag{3.6}$$

Relationships (3.4)-(3.6) and (1.13)-(1.15) constitute a closed system of equations that determine the synthesis of control in problem (3.1), (1.4) and, consequently, also the formulas of optimal interpolation. Similarly to the proof of Theorem 1 we establish that, when its requirements relative to coefficients of formulas (1.8) and (1.9) are satisfied, the problem defined by (3.4)-(3.6) and (1.13)-(1.15) has a unique solution.

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Translated by J.J. D.